Specialist Mathematics Unit 3 & 4 — Examination 1 (Practice 1) — Answers & Explanations

Answers to the 5 questions in the free preview.

Examination 1 — technology-free

1. (3 marks in total)

True for $n = 1$ since $25 - 1 = 24$. If $25^k - 1 = 24m$, then $25^{k+1} - 1 = 24(25m + 1)$, so the statement holds for all $n \in \mathbb{N}$ by induction.

[3 marks]

Let $P(n)$ be the statement "$25^n - 1$ is divisible by 24". Base case: $25^1 - 1 = 24$, so $P(1)$ is true. Inductive step: assume $P(k)$, so $25^k - 1 = 24m$ for some $m \in \mathbb{Z}$. Then $25^{k+1} - 1 = 25 \times 25^k - 1 = 25(24m + 1) - 1 = 24 \times 25m + 24 = 24(25m + 1)$, which is divisible by 24, so $P(k) \Rightarrow P(k + 1)$. Since $P(1)$ is true, $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction. Marks: 1 for the base case; 1 for using the assumption to write $25^{k+1} - 1$ as a multiple of 24; 1 for a complete conclusion. Common error: "assuming what is to be proved" by starting from $25^{k+1} - 1 = 24p$.

2. (4 marks in total)

a. $z = 2\,\mathrm{cis}\left(\frac{2\pi}{3}\right)$

[1 mark]

$|z| = \sqrt{1 + 3} = 2$. z is in the second quadrant with $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$, so $\mathrm{Arg}(z) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Common error: giving $-\frac{\pi}{3}$ from $\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$ without checking the quadrant.

b. $n = 3$, $z^3 = 8$

[1 mark]

By de Moivre's theorem $z^n = 2^n\,\mathrm{cis}\left(\frac{2n\pi}{3}\right)$, which is real and positive when $\frac{2n\pi}{3}$ is a multiple of 2π, first at $n = 3$: $z^3 = 8\,\mathrm{cis}(2\pi) = 8$.

c. $z = 2$, $z = -1 + \sqrt{3}i$, $z = -1 - \sqrt{3}i$

[2 marks]

From part b., $-1 + \sqrt{3}i$ is a solution, and so is its conjugate $-1 - \sqrt{3}i$ (the coefficients are real); $z = 2$ is the real solution. Equivalently $z^3 - 8 = (z - 2)(z^2 + 2z + 4)$ and $z^2 + 2z + 4 = 0$ gives $z = -1 \pm \sqrt{3}i$. 1 mark for the method (conjugate root theorem, factorising, or $z = 2\,\mathrm{cis}\left(\frac{2k\pi}{3}\right)$); 1 mark for all three solutions.

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3. (5 marks in total)

a. $x^2 + 3 = (x-1)(x+1) + 4$, so $f(x) = x + 1 + \dfrac{4}{x-1}$.

[1 mark]

Write the numerator as $(x^2 - 1) + 4 = (x-1)(x+1) + 4$ and divide each term by $x - 1$. Working backwards is also acceptable: $x + 1 + \frac{4}{x-1} = \frac{(x+1)(x-1)+4}{x-1} = \frac{x^2+3}{x-1}$.

b. $(-1, -2)$ and $(3, 6)$

[2 marks]

$f'(x) = 1 - \dfrac{4}{(x-1)^2} = 0$ gives $(x-1)^2 = 4$, so $x = -1$ or $x = 3$. Then $f(-1) = \frac{4}{-2} = -2$ and $f(3) = \frac{12}{2} = 6$. 1 mark for the x-values, 1 mark for both coordinates.

c. Vertical asymptote $x = 1$; oblique asymptote $y = x + 1$; local maximum $(-1, -2)$; local minimum $(3, 6)$; y-intercept $(0, -3)$; no x-intercepts.

[2 marks]

The branch for $x < 1$ rises to the local maximum $(-1, -2)$, passes through $(0, -3)$ and falls to $-\infty$ as $x \to 1^-$; the branch for $x > 1$ comes down from $+\infty$ to the local minimum $(3, 6)$ and then approaches $y = x + 1$ from above. There are no x-intercepts because $x^2 + 3 > 0$. 1 mark for both asymptotes with equations; 1 mark for the correct shape with the labelled points.

4. (3 marks in total)

$y = 1 - 2x$

[3 marks]

Differentiating implicitly with respect to x: $\dfrac{1}{1+y^2}\dfrac{dy}{dx} + y^2 + 2xy\dfrac{dy}{dx} = 0$. At $(0, 1)$: $\dfrac{1}{2}\dfrac{dy}{dx} + 1 = 0$, so $\dfrac{dy}{dx} = -2$. The tangent is $y - 1 = -2(x - 0)$, that is, $y = 1 - 2x$. Marks: 1 for correct implicit differentiation (including the product rule on xy^2); 1 for the gradient; 1 for the equation. Common error: differentiating xy^2 as $2y\dfrac{dy}{dx}$.

5. (3 marks in total)

$\dfrac{\pi}{4} - \dfrac{1}{2}\log_e(2)$

[3 marks]

Integrate by parts with $u = x$ and $\dfrac{dv}{dx} = \sec^2(x)$, so $v = \tan(x)$:

$$\int_0^{\frac{\pi}{4}} x\sec^2(x)\,dx = \Big[x\tan(x)\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan(x)\,dx = \frac{\pi}{4} + \Big[\log_e(\cos(x))\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \log_e\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} - \frac{1}{2}\log_e(2).$$

Marks: 1 for the integration by parts; 1 for $\int \tan(x)\,dx = -\log_e(\cos(x))$; 1 for the answer in the required form ($a = 4$, $b = 2$, $c = 2$).

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PREVIEW

That’s the end of the preview answers

These are the answers to the 5 preview questions. The full answer key covers all 10 questions in Specialist Mathematics Unit 3 & 4 — Examination 1 (Practice 1), with the working behind every mark.

STILL TO COME	QUESTIONS	MARKS
Examination 1 — technology-free	5	22

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