# Specialist Mathematics Unit 3 & 4 — Examination 1 (Practice 1) — Answers & Explanations

For a parent, tutor, or the student to mark the exam paper against.

## Examination 1 — technology-free

**1.** (3 marks in total)

**a.** Left side = 1, right side = $1^2 = 1$. [1 mark]

Both sides equal 1, so the statement holds for $n = 1$.

**b.** $1 + 3 + \ldots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2$. [2 marks]

Add the next term, $2(k + 1) - 1 = 2k + 1$, to both sides of the assumption. The result is $(k + 1)^2$, so the statement is true for $n = k + 1$ whenever it is true for $n = k$; with part a, it is true for all $n \in N$ by induction. One mark for using the assumption, one for the conclusion.

**2.** (4 marks in total)

**a.** $z = 2\text{cis}(-\pi/3)$ [1 mark]

$|z| = \sqrt{(1 + 3)} = 2$ and $\text{Arg}(z) = -\pi/3$ because z is in the fourth quadrant with $\tan^{-1}(\sqrt{3}) = \pi/3$.

**b.** 64 [1 mark]

By De Moivre's theorem, $z^6 = 2^6\text{cis}(-2\pi) = 64$.

**c.** $\sqrt{2}\text{cis}(-\pi/6)$ and $\sqrt{2}\text{cis}(5\pi/6)$ [2 marks]

Square roots have modulus $\sqrt{2}$ and arguments $(-\pi/3)/2 = -\pi/6$ and $-\pi/6 + \pi = 5\pi/6$. One mark for each root.

**3.** (5 marks in total)

**a.** $f(x) = x + 1 + 2/(x - 1)$ [1 mark]

$x^2 + 1 = (x - 1)(x + 1) + 2$.

**b.** $x = 1$ and $y = x + 1$ [2 marks]

The vertical asymptote comes from $x - 1 = 0$; as $x \to \pm\infty$, $2/(x - 1) \to 0$, so $y = x + 1$ is an oblique asymptote. One mark for each.

**c.** $(1 + \sqrt{2}, 2 + 2\sqrt{2})$ and $(1 - \sqrt{2}, 2 - 2\sqrt{2})$ [2 marks]

$f'(x) = 1 - 2/(x - 1)^2 = 0$ gives $(x - 1)^2 = 2$, so $x = 1 \pm \sqrt{2}$. Then $f(1 + \sqrt{2}) = 2 + \sqrt{2} + 2/\sqrt{2} = 2 + 2\sqrt{2}$ and $f(1 - \sqrt{2}) = 2 - 2\sqrt{2}$. One mark for the x-values, one for the coordinates.

**4.** (4 marks in total)

**a.** 1/4 [2 marks]

$du = -\sin(x)\,dx$, and the terminals become $u = 1$ and $u = 0$: $\int u^3\,du$ from 0 to 1 = 1/4. One mark for the transformed integral, one for the value.

**b.** $(x/2)e^{2x} - (1/4)e^{2x} + c$ [2 marks]

Let $u = x$ and $dv = e^{2x}\,dx$, so $v = (1/2)e^{2x}$. Then $\int xe^{2x}\,dx = (x/2)e^{2x} - \int (1/2)e^{2x}\,dx$. One mark for the set-up, one for the answer.

**5.** (4 marks in total)

**a.** $y = 3e^{(x^2)}$ [3 marks]

Separate: $\int (1/y)\,dy = \int 2x\,dx$, so $\log_e|y| = x^2 + c$ and $y = Ae^{(x^2)}$. $y(0) = 3$ gives $A = 3$. One mark for separating, one for integrating, one for the constant.

**b.** 6 [1 mark]

$y = 3e^{(\log_e(2))} = 3 \times 2 = 6$.

**6.** (4 marks in total)

**a.** $-4$ [1 mark]

$2 \times 1 + (-1) \times 2 + 2 \times (-2) = 2 - 2 - 4 = -4$.

**b.** $\cos^{-1}(-4/9)$ [1 mark]

$|a| = |b| = 3$, so $\cos(\theta) = -4/9$.

**c.** $-2i + 6j + 5k$ [2 marks]

$a \times b = ((-1)(-2) - 2 \times 2)i - (2 \times (-2) - 2 \times 1)j + (2 \times 2 - (-1) \times 1)k = -2i + 6j + 5k$. Check: it is perpendicular to a and b (dot products $-4 - 6 + 10 = 0$ and $-2 + 12 - 10 = 0$). One mark for method, one for the vector.

**7.** (4 marks in total)

**a.** 1  [1 mark]

$2 \times 5 - 3 \times 3 = 10 - 9 = 1$.

**b.** 97  [2 marks]

For independent variables, $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$: $4 \times 4 + 9 \times 9 = 16 + 81 = 97$. One mark for squaring the coefficients, one for adding the variances.

**c.** 1/2  [1 mark]

$\mathrm{sd}(\bar{X}) = \sigma/\sqrt{n} = 2/\sqrt{16} = 1/2$.

**8.** (4 marks in total)

**a.** −4/5  [3 marks]

$2x + y + x(dy/dx) + 2y(dy/dx) = 0$, so $dy/dx = -(2x + y)/(x + 2y)$. At (1, 2): $-4/5$. One mark for differentiating xy, one for rearranging, one for the value.

**b.** $y = -(4/5)x + 14/5$  [1 mark]

$y - 2 = -(4/5)(x - 1)$.

**9.** (4 marks in total)

**a.** $P(2) = 8 - 8 + 18 - 18 = 0$  [1 mark]

By the factor theorem.

**b.** $z = 2, 3i, -3i$  [2 marks]

$P(z) = (z - 2)(z^2 + 9)$, and $z^2 + 9 = 0$ gives $z = \pm 3i$. One mark for the quadratic factor, one for the roots.

**c.** 6  [1 mark]

The vertices are 2, 3i and −3i: the base along the imaginary axis is 6 and the height is 2, so the area is 6.

**10.** (4 marks in total)

**a.** 8π  [2 marks]

$V = \pi\int x\,dx$ from 0 to 4 $= \pi \times 8 = 8\pi$. One mark for the integral, one for the value.

**b.** π/4  [2 marks]

$\int 1/(1 + x^2)\,dx$ from 0 to 1 $= \tan^{-1}(1) - \tan^{-1}(0) = \pi/4$. One mark for the antiderivative, one for the value.